Jason Slowbe

Pi Filling, Archimedes Style

A lesson on finding the area of regular polygons using the formula

$$A = (1/2) \text{ (apothem)(perimeter)}$$

can provide a rich exploration of the relationship between inscribed polygons and the development of an algorithm that systematically generates the digits of π.

In the spirit of Archimedes' method of approximating π, we will inscribe regular polygons in the unit circle, which is known to have area π. As the number of sides, n, in each regular polygon increases, the polygons fill more and more of the area inside the unit circle. Naturally, increasing n causes the areas of the n-gons to approach π. With sufficiently large n, we can obtain decimal approximations that become arbitrarily close to the exact value of π. Of course, a problem arises with large n, as the calculations quickly become unbearably tedious without the use of a calculator. Fortunately, this process can be automated very easily by writing a program to compute the area for any n-gon, thus providing a compelling exploration of a method for generating the never-ending digits of π.

CALCULATING AREAS

First, consider the case of a regular (equilateral) triangle inscribed in the unit circle. The distance from the center to any vertex is equal to the circle's radius. Using the well-known formula for the sum of the interior angles, S, of any n-gon, we have

$$S = (n - 2) \cdot 180.$$

Therefore, each angle of the regular triangle measures

$$\frac{(3-2)180}{3} = 60°$$

See **figure 1**.

Now, consider the isosceles triangle formed by two radii and a side of the equilateral triangle. The radii bisect the angles to which they are drawn, so the base angles of the isosceles triangle measure

$$\frac{(n-2)\cdot 180}{2n} = \frac{(3-2)180}{2\cdot 3}$$
$$= 30°.$$

See **figure 2**.

We label the apothem y and the bases of each right triangle x. Using trigonometry, we can find the lengths of the apothem, y, and the base, $2x$. The base will then be multiplied by n, the number of sides, to obtain the perimeter. Finally, we will be able to use $A = (1/2)(\text{apothem})(\text{perimeter})$ to find the area of the regular triangle. Solve for x and y, as shown in **figure 3**.
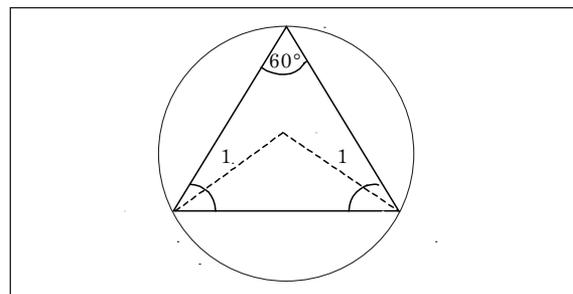

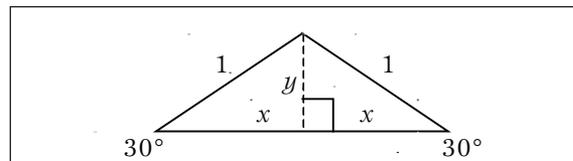**Fig. 1**

Fig. 2

$$\cos(30°) = x/1 \qquad \sin(30°) = y/1$$
$$\text{base} = 2x = 2\cos(30°) \qquad \text{apothem} = \sin(30°)$$

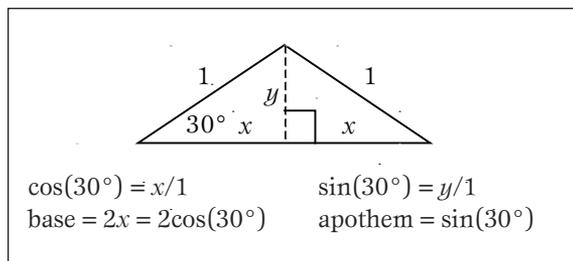**Fig. 3**

With the apothem = $\sin(30°)$ and the perimeter = $3 \cdot 2 \cos(30°)$, we can calculate the area of the regular triangle in the following way:

$$
\begin{aligned}
\text{Area} &= (1/2)\,(\text{apothem})(\text{perimeter})\\
&= [1/2][\sin(30°)][3 \cdot 2 \cos(30°)]\\
&= [1/2][\sin(30°)][3 \cdot 2 \cos(30°)]\\
&\approx 1.2990 \text{ sq. units} \qquad\qquad (1)
\end{aligned}
$$

Following this same process for a square, we get:

$$
\begin{aligned}
\text{Area} &= (1/2)(\text{apothem})(\text{perimeter})\\
&= [1/2][\sin(45°)][4 \cos(45°)]\\
&= [1/2][\sin(45°)][4 \cdot 2 \cos(45°)]\\
&= 2.0000 \text{ sq. units} \qquad\qquad (2)
\end{aligned}
$$

See **figure 4**.

In fact, we quickly notice that equations (1) and (2) are remarkably similar. The number of sides n has changed, as have the degree measures of each base angle in the isosceles triangle. But those base angles are dependent on n. In particular, the degree measure of the base angles of the inner isosceles triangles can be found by finding the total measure of the interior angles of the regular n-gon, dividing this sum by n to get the measure of each angle, and dividing that angle by 2 to account for the bisection by the radius. So, we can generalize the area of the regular n-gons inscribed in the unit circle by the following:

$$
\text{Area} = \left[1/2\right]\left[\sin\left(\frac{(n-2)\cdot 180}{2n}\right)\right]
$$
$$
\cdot\left[n \cdot 2\cos\left(\frac{(n-2)\cdot 180}{2n}\right)\right] \quad (3)
$$

We now want to program a graphing calculator to complete these calculations for any value of n specified by the user. Based on your students' programming experiences, you can provide varying levels of guidance to challenge each student appropriately. For purposes of this article, I used a TI-83, but any calculator with programming capabilities can be used as well.

One way to introduce and motivate programming for inexperienced students is to store a particular numeric value for the number of sides of the polygon into the calculator, then enter the area expression just as it
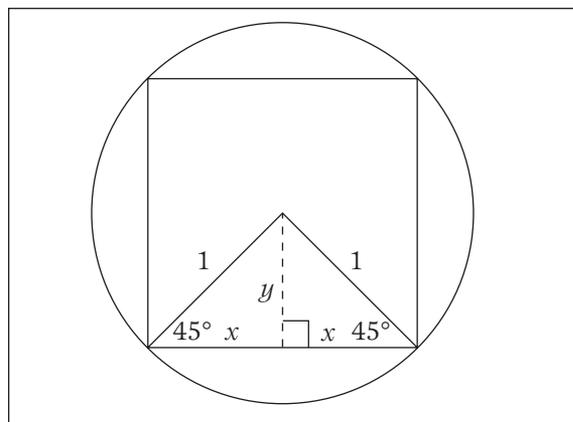

Fig. 4

Fig. 5

Fig. 6

appears in (3). The variable N is a "memory location" in the calculator, where numeric values can be stored and then substituted in subsequent calculations.

The downside to such an approach is the need to re-store new values of n for each polygon, then recall the long expression to calculate the area of each n-gon. A program can make this process a bit easier (**fig. 5**). The user inputs the desired number of sides n, the calculator computes the area and stores this value into the variable A, and the calculator displays the number of sides N and its corresponding area A.
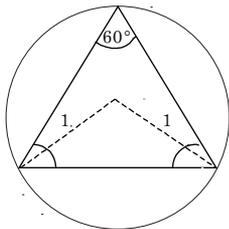
GENERATING π

As n increases, the areas displayed by the program will approach 3.141592654 (the maximum num-
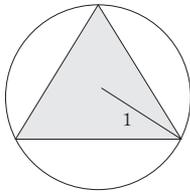
As we increase the number of sides in regular polygons inscribed in the unit circle, what happens to their areas?
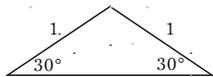
Example: Triangle
(a) The goal is to find the area of the regular (equilateral) triangle, represented below.

(b) Consider the isosceles triangle below, formed by drawing radii to two of the vertices of the triangle. Note that the base angles of the isosceles triangle are half the angles of the regular triangle. These base angles can be represented as

$$\frac{1}{2} \cdot \frac{(n-2)(180)}{n} = \frac{(n-2)(180)}{2n}.$$

(c) Next, form a right triangle by drawing the apothem from the center of the circle perpendicular to the base (see figure below). Using sine and cosine, we can find the lengths of the apothem, y, and the base, $2x$. Use degree mode.
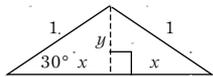
$$\cos(30°) = x/1 = x \qquad \sin(30°) = y/1 = y$$
$$\text{base} = 2x = 2\cos(30°) \qquad \text{apothem} = \sin(30°)$$

(d) We now have enough information to calculate the area of the regular triangle.

Area = (1/2)(perimeter)(apothem)
  = [1/2] [2cos(30°) + 2cos(30°)
    + 2cos(30°)] [sin(30°)]
  = [1/2] [3 · 2cos(30°)][sin(30°)]
  ≈ 1.2990 sq. units

Fig. 7 Example: Triangle

ber of digits displayed on the viewing screen; the last digit, 4, is rounded up from 3). With a minor accommodation, we can have the program automatically display areas for increasing values of n (**fig. 6**). The program starts with $n = 3$, displays the number of sides and the corresponding area, pauses to allow the user time to see the results on the screen, then adds 1 to n and returns to the Lbl B line, repeating the process for $n = 4, 5, 6$, and so on.

An interesting question to explore is "After what value of n do the digits of π stabilize in the hundredths place value?" This does not occur until $n = 114$, indicating that this algorithm for generating the digits of π is not very efficient. To improve its efficiency, incrementing n by a value larger than 1 will allow the user to see the areas of large n-gons without needing to hit Enter many times. The second-to-last line of the program would be changed to ":10+N→ N" in order to increment n by 10, for example. The calculator would then display the areas of the 3-gon, 13-gon, 23-gon, and so on.

CONCLUSION

Graphing calculators are widely used in high school mathematics classrooms, but their programming capabilities are underused. These capabilities can transform an ordinary geometry lesson into a rich mathematical exploration that is interesting and accessible to students with a wide range of abilities. One of the most beneficial aspects of this lesson is teaching students to extend and generalize mathematical results, which is an important skill for students to learn as they begin studying more abstract mathematical ideas. As well, training students to think more broadly and universally helps them develop critical thinking skills and build a strong foundation for future mathematical learning. The lesson helps to satisfy student curiosity regarding the irrational nature of π, making use of the graphing calculator to develop an infinite algorithm that makes the "never-ending" nature of π more apparent.

SOLUTIONS

Figure 7 shows a complete solution for a regular inscribed triangle. The worksheet itself provides response questions for students for the case of the triangle and more advanced cases as well. Depending on your students, you may want to give them the triangle solution so they can reference it for other n-gons. Or, to leave the problem open-ended, you can copy only the worksheet for students, then provide whatever guidance or assistance is most appropriate for the needs of your students.

1. See **table 1**.

| Table 1												
Answer to Question 1												
n	3	4	5	6	7	8	9	10	...	50	...	200
Area	1.2990	2.0000	2.3776	2.5981	2.7364	2.8284	2.8925	2.9389	...	3.1333	...	3.1411

2. The areas will approach the area of the unit circle, $\pi(1)^2 = \pi$.

3. Circumscribing regular polygons around the unit circle will overestimate the value of π. Therefore, using these methods with circumscribed polygons will generate areas that approach π from above.

4. See **figure 5** for code that allows users to input any specific value for n. See **figure 6** for code that starts with $n = 3$ and increments n by a determined value each time the user hits Enter. ∞

JASON SLOWBE, slowbe@strongnet
.org, teaches mathematics at Strongs-
ville High School, Cleveland, OH 44149.
He is interested in innovative uses of
technology for extending the curriculum and mak-
ing mathematics accessible for all students.
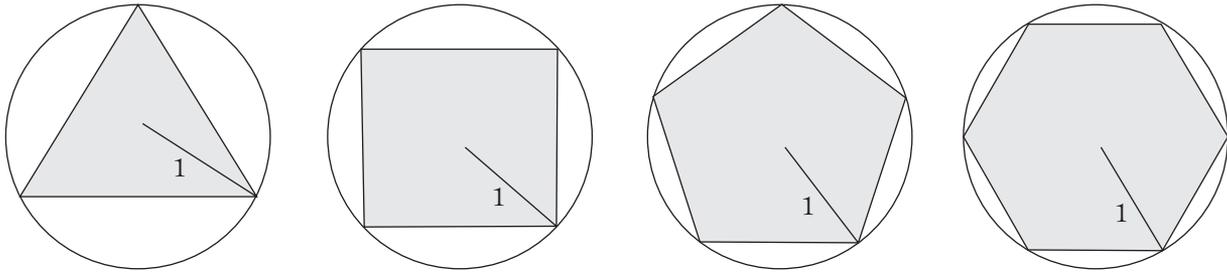
As the number of sides in a regular polygon inscribed in the unit circle increases, what happens to the area of the polygon?

1. Calculate the area of each regular *n*-gon inscribed in the unit circle rounded to the nearest ten-thousandth. Use calculator values of sine and cosine throughout the problem, and round only in the last step. The areas of the regular triangle and quadrilateral are provided to check your work.

n	3	4	5	6	7	8	9	10	…	50	…	200
Area	1.2990	2.0000							…		…	

2. As *n* increases, what number should the areas approach? Explain.

3. What would be different about the areas if the regular polygons were circumscribed around the unit circle? Explain.
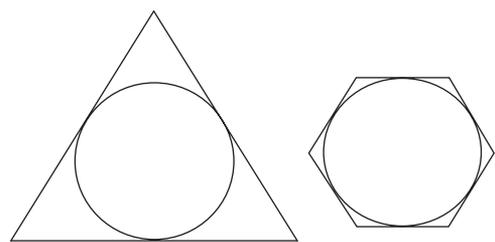
4. (Challenge) Write a calculator program to generalize your results. That is, with input *n,* the program should output the area of the *n*-gon.